On Analysis of Kernel Collocation Methods for Spherical PDEs

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Abstract

In this paper the error analysis of the kernel collocation method for partial differential equations on the unit sphere is presented. A simple analysis is given when the true solutions lie in arbitrary Sobolev spaces. This also extends the previous studies for true solutions outside the associated native spaces. Finally, some experimental results support the theoretical error bounds.

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1 Introduction

Kernels are widely used for fitting a surface to scattered data arising from sampling an unknown function defined on $\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} : \|x\|_2 = 1\}$. This is a basic building block for constructing meshfree numerical methods for solving partial differential equations (PDEs) on $\mathbb{S}^d$ which is an area of growing interest with applications to physical geodesy, potential theory, oceanography, and meteorology. Among all numerical methods for PDEs, the *collocation method* which samples the input operator at a predetermined set of scattered points, appears to be the simplest one. We address the Galerkin and the Petrov-Galerkin methods as other well-established methods for discretizing and solving some spherical PDEs [10, 16, 13]. The major advantage of collocation method over Galerkin-type methods is the minimization of the computational effort with respect to quadrature, since for each degree of freedom only one point evaluation at a so-called collocation point is required. This property constitutes a significant advantage for applications on surfaces where constructing an efficient quadrature and forming the stiffness matrices are rather costly.

We should mention that the RBF collocation method for solving PDEs on bounded Euclidian domains has a rather longer history, starting from a couple of papers by Kansa for problems in fluid dynamics [8, 9]. The approach of Kansa is also called the *unsymmetric* RBF collocation method and has been widely used for many types of PDEs thereafter. However, it was shown in [7] that this method may lead to nonsolvable systems in some rare situations. By changing the setting to a least squares approximation, a general error analysis for unsymmetric collocation methods in provided in [23, 24]. On the other hand, the *symmetric* RBF collocation method, known also as *generalized Hermite interpolation*, was introduced in [28, 19]. The first error bounds were obtained in [2, 3]. See also [5, 26, 27] for some newer sources.

In this paper we analyze the numerical solution of PDEs on the unit sphere by collocation at scattered data points with restricted kernels. The restriction of a positive definite kernel from $\mathbb{R}^{d+1}$ to any submanifold (such as $\mathbb{S}^d$) is a simple way for obtaining a positive definite kernel on the submanifold [4]. In [12, 14] the collocation method based on the generalized Hermite interpolation on the sphere is investigated while in [11] both standard and Hermite based methods are studied. If the PDE is posed on the whole sphere without boundary conditions then both methods have the same structure with different trial kernels. Thus the error analysis and stabilities are the same. In the above sources the error bounds have been provided for PDEs with true solutions in some specific Sobolev (native) spaces. In [21], the error analysis is performed for the Galerkin method and the error estimate for collocation method is obtained from that for the Galerkin method. The analysis covers a wider range of Sobolev spaces smoother than the native space of the trial kernel.

In this work we give an analysis special for the collocation method which on the one hand seems to be simpler and on the other hand extends the previous analysis for true solutions come from arbitrary Sobolev spaces either smoother or coarser than the native spaces.

The organization of the paper is as follows. In section 2 we review some tools for approximation theory on the unit sphere. In section 3 the kernel collocation method is discussed. The error analysis is given in section 4, and the experimental results are reported in section 5.

2 Basis functions and spaces on the unit sphere

In this section some preliminary results about polynomials on the sphere and about *spherical basis functions (SBFs)* are reviewed. Then the Sobolev spaces associated to some SBFs are briefly addressed. The section will end by a short description about a class of *restricted kernels* on the unit sphere.

We start with *spherical harmonics* which are basic tools for approximation theory on the sphere. Spherical harmonics are restrictions to the unit sphere $\mathbb{S}^d$ of polynomials Y which satisfy $\Delta Y = 0$, where Δ is the Laplacian operator in $\mathbb{R}^{d+1}$. The space of all spherical harmonics of degree ℓ on $\mathbb{S}^d$ is denoted by $\mathcal{H}_\ell^d$, and has an L_2 orthonormal basis

$$\big\{Y_{\ell k} : k = 1, \dots, N(d, \ell)\big\},$$

where

$$N(d,0) = 1, \quad N(d,\ell) = \frac{(2\ell + d - 1)\Gamma(\ell + d - 1)}{\Gamma(\ell + 1)\Gamma(d)}, \quad \ell \geqslant 1,$$

where Γ is the known Gamma function. The space of spherical harmonics of order m or less will be denoted by

$$\mathcal{P}_m^d := \bigoplus_{\ell=0}^{m} \mathcal{H}_\ell^d,$$

with dimension $N(d + 1, m)$. It is known that the spherical harmonics are the eigenfunctions of the Laplace-Beltrami operator Δ_0, and every function $u \in L_2 = L_2(\mathbb{S}^d)$ can be expanded as

$$u = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} \widehat{u}_{\ell k} Y_{\ell k}, \quad \widehat{u}_{\ell k} = \frac{1}{\omega_d} \int_{\mathbb{S}^d} Y_{\ell k} u \, d\sigma.$$

where ω_d denotes the surface area of $\mathbb{S}^d$ and $d\sigma$ is the surface measure of the unit sphere. The L_2-norm of u given by the formula

$$\|u\|_0^2 := \int_{\mathbb{S}^d} |u|^2 d\sigma,$$

can also be expressed, via Parseval's identity, as

$$\|u\|_0^2 = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} |\widehat{u}_{\ell k}|^2.$$

Finally, we note that

$$\sum_{k=1}^{N(d,\ell)} Y_{\ell k}(x) Y_{\ell k}(y) = \frac{N(d,\ell)}{\omega_d} P_\ell(d + 1; x^T y), \tag{2.1}$$

is the known *addition formula* for spherical harmonics. We refer the reader to [1, 15] for more details.

Next, we explain the approximation space used in this paper. This space is formed via so-called *zonal kernels*.

Definition 2.1 *A kernel $\Phi : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ is called zonal if $\Phi(x, y) = \phi(x^T y)$ for all $x, y \in \mathbb{S}^d$, where ϕ is a continuous univariate function on $[-1, 1]$.*

We are specially interested in zonal kernels of the type

$$\Phi(x, y) = \phi(x^T y) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(d+1; x^T y), \quad a_\ell > 0, \ \sum_{\ell=0}^{\infty} a_\ell < \infty,$$

where $\{P_\ell(d+1; t)\}_{\ell=0}^{\infty}$ is the sequence of $(d+1)$-dimensional Legendre polynomials normalized to $P_\ell(d+1; 1) = 1$. In [25] and [29] it was proved that such ϕ is positive definite on $\mathbb{S}^d$. Using the addition formula (2.1), the kernel $\Phi(x, y)$ may also be expressed in terms of spherical harmonics as

$$\Phi(x, y) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} \widehat{\phi}(\ell) Y_{\ell k}(x) Y_{\ell k}(y), \tag{2.2}$$

where

$$\widehat{\phi}(\ell) = \frac{\omega_d}{N(d, \ell)} a_\ell.$$

If we assume that for some $\sigma > d/2$,

$$c_\Phi (1 + \ell)^{-2\sigma} \leqslant \widehat{\phi}(\ell) \leqslant C_\Phi (1 + \ell)^{-2\sigma}, \quad \ell \geqslant 0, \tag{2.3}$$

holds for specific positive constants c_Φ and C_Φ, then the *native space* associated to Φ is norm equivalent to $H^\sigma = H^\sigma(\mathbb{S}^d)$, the Sobolev space of order σ on $\mathbb{S}^d$. If fact, the native space $\mathcal{N}_\Phi = \mathcal{N}_\Phi(\mathbb{S}^d)$ is defined by

$$\mathcal{N}_\Phi := \Big\{ u \in \mathcal{D}'(\mathbb{S}^d) : \|u\|_\Phi^2 := \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} \frac{|\widehat{u}_{\ell k}|^2}{\widehat{\phi}(\ell)} < \infty \Big\},$$

where $\mathcal{D}'(\mathbb{S}^d)$ is the space of distributions on $\mathbb{S}^d$. It can be shown that $\mathcal{N}_\Phi$ is a Hilbert space with respect to the inner product

$$\langle u, v \rangle_\Phi := \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} \frac{\widehat{u}_{\ell k} \widehat{v}_{\ell k}}{\widehat{\phi}(\ell)}, \quad u, v \in \mathcal{N}_\Phi.$$

Moreover, Φ is reproducing kernel for $\mathcal{N}_\Phi$, i.e., for all $u \in \mathcal{N}_\Phi$,

$$\langle u, \Phi(x, \cdot) \rangle_\Phi = u(x), \quad x \in \mathbb{S}^d.$$

On the other hand, the Sobolev space H^σ with real parameter σ is defined by

$$H^\sigma = H^\sigma(\mathbb{S}^d) := \Big\{ u \in \mathcal{D}'(\mathbb{S}^d) : \|u\|_\sigma^2 := \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} (1+\ell)^{2\sigma} |\widehat{u}_{\ell k}|^2 < \infty \Big\}.$$

Thus, under condition $\sigma > d/2$, due to the definitions of $\mathcal{N}_\Phi$ and H^σ and condition (2.3), we have

$$c_\Phi \|u\|_\Phi^2 \leqslant \|u\|_\sigma^2 \leqslant C_\Phi \|u\|_\Phi^2,$$

which means that $\mathcal{N}_\Phi$ and H^σ are norm equivalent.

A class of basis functions which satisfy condition (2.3) for some σ can be obtained by restricting the positive definite kernels from $\mathbb{R}^{d+1}$ to $\mathbb{S}^d$. If the original kernel is positive definite, so is its restriction to $\mathbb{S}^d$, making it well-suited for scattered data interpolation problems on $\mathbb{S}^d$. This type of kernels has been studied in [20, 17, 30] while the general case, concerning an arbitrary submanifold, has been investigated in [4]. Let's make it more precise. Assume that S is a radial basis function on $\mathbb{R}^{d+1}$, i.e. there exists a univariate function ψ such that $S(x) = \psi(\|x\|_2)$ where $\|\cdot\|_2$ is Euclidian norm in $\mathbb{R}^{d+1}$. Since for points $x, y \in \mathbb{S}^d$ we have $\|x-y\|_2 = \sqrt{2-2x^Ty}$, we may therefore define

$$\Phi(x,y) = \phi(x^Ty) := \psi(\sqrt{2-2x^Ty}) = \psi(\|x-y\|_2) = S(x-y), \quad x, y \in \mathbb{S}^d.$$

It is clear that Φ inherits the property of positive definiteness from S. In [17] it was proved that if Φ is represented in the form (2.2) then the Fourier coefficients $\widehat{\phi}(\ell)$ satisfy the decay condition (2.3) for some $\sigma > 0$. To be more precise, if we assume that the radial basis function S has $H^s(\mathbb{R}^{d+1})$ as its native space, which is equivalent to this fact that its $(d+1)$-variate Fourier transform $\widehat{S}(\omega)$ behaves like

$$(1 + \|\omega\|_2^2)^{-s}, \quad \omega \in \mathbb{R}^{d+1},$$

for $s > \frac{d+1}{2}$, then Φ (the restriction of S on $\mathbb{S}^d$) generates $H^{s-1/2}(\mathbb{S}^d)$, i.e. its Fourier coefficients satisfy (2.3) for $\sigma = s - \frac{1}{2}$. In the general case, when S is restricted to a k-dimensional smooth submanifold $\mathbb{M}^k \subset \mathbb{R}^{d+1}$, then the native space of the restricted kernel is $H^{s-(d+1-k)/2}(\mathbb{M}^k)$. See [4, Theorem 5].

3 Collocation method

We consider a general PDE problem

$$Lu = f \text{ on } \mathbb{S}^d, \tag{3.1}$$

where L is a self-adjoint differential operator of order κ, for some $\kappa > 0$, and f is a given known right hand side function. The unknown function u is assumed to lie in a proper Sobolev space which will be determined explicitly below. In additions, we assume that Lu is expressed as a Fourier series

$$Lu = \sum_{k=1}^{\infty} \sum_{k=1}^{N(d,\ell)} \widehat{L}(\ell)\widehat{u}_{\ell k} Y_{\ell k},$$

in which

$$c_L(1+\ell)^{\kappa} \leqslant \widehat{L}(\ell) \leqslant C_L(1+\ell)^{\kappa}, \quad \ell \geqslant 0, \tag{3.2}$$

where c_L, C_L are two positive constants independent of ℓ. This means that L is *strongly elliptic*. For example, for special case $L = -\Delta_0 + \omega^2 I$ we have $\widehat{L}(\ell) = \ell(\ell+d-1) + \omega^2$ and $\kappa = 2$. Although the method can be successfully applied on even more general cases, our assumptions above pave the way to analyze the method, perfectly.

Assume that Φ is a kernel that satisfies condition (2.3) for some $\sigma > d/2$. Suppose

$$X = \{x_1, x_2, \ldots, x_N\} \subset \mathbb{S}^d$$

is a given discrete set of scattered points on the unit sphere $\mathbb{S}^d$. The numerical solution

$$u_N = u_{N,\Phi,X} = \sum_{j=1}^{N} b_j \Phi(\cdot, x_j)$$

from the trial space

$$V_{\Phi,X} := \operatorname{span}\{\Phi(\cdot, x_j) : x_j \in X\}$$

is simply determined by collocation conditions

$$Lu_N(x_k) = f(x_k), \quad k = 1, \ldots, N. \tag{3.3}$$

This leads to the linear system

$$Ab = F, \tag{3.4}$$

where $A = (a_{kj})$ is a $(N \times N)$-matrix with $a_{kj} = (L\Phi(\cdot, x_j))(x_k)$, $k, j = 1, \ldots, N$, and $F = (f_k)$ is a N-vector with $f_k = f(x_k)$, $k = 1, \ldots, N$.

If Φ is a zonal kernel represented by the Fourier series (2.2), then $L\Phi$ is a zonal kernel having the Fourier expansion

$$L\Phi(x,y) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} \widehat{\phi}(\ell)\widehat{L}(\ell) Y_{\ell k}(x) Y_{\ell k}(y). \tag{3.5}$$

The collocation matrix A is formed via kernel

$$\Lambda(x,y) := (L\Phi(\cdot, y))(x)$$

which is a positive definite kernel provided that Φ is positive definite, because from (3.5) the Fourier coefficients of kernel $\Lambda(x,y)$ are $\widehat{\lambda}(\ell) = \widehat{\phi}(\ell)\widehat{L}(\ell)$. Thus we have the following Theorem from [11]. Before that, to shorten our presentation we collect global assumptions on differential operator L and trial kernel Φ that we will employ almost throughout the rest of the paper.

Assumption 3.1 *Let L be a self-adjoint differential operator satisfying* (3.2) *for positive integer κ, and Φ be a positive definite spherical basis function on $\mathbb{S}^d$ satisfying* (2.3) *for $\sigma > d/2 + \kappa/2$.*

Theorem 3.2 *Under the Assumption 3.1, there exists a unique function $u_N \in V_{\Phi,X}$ that fulfills the conditions* (3.3)*.*

Since the final linear system is positive definite (and sparse for compactly supported kernels), special and fast linear algebra solvers can be used in practical situations.

4 Error Analysis

This section is devoted to convergence analysis of the collocation method of preceding section. The final error bounds are applicable for solution u in Sobolev spaces H^γ for $\kappa \leqslant \gamma \leqslant 2\sigma$ where κ is the order of L and σ is the smoothness index of kernel Φ which is determined by (2.3). The errors are measured in Sobolev norms $\|\cdot\|_\beta$ for $\kappa \leqslant \beta \leqslant \min\{\gamma, \sigma + \kappa/2\}$ and $\beta = 0$.

At the starting point, we express the relation between appropriate norms of u and Lu for differential operator L. Although this is a well known property of Sobolev spaces and is also mentioned in passing in [11], we bring a proof for readers conveniences.

Theorem 4.1 *If L satisfies* (3.2) *for a positive constant κ and $u \in H^\tau$ for $\tau \geqslant \kappa$, then $Lu \in H^{\tau-\kappa}$ and there exist constants c and C such that*

$$c\,\|u\|_\mu \leqslant \|Lu\|_{\mu-\kappa} \leqslant C\,\|u\|_\mu \tag{4.1}$$

hold for any μ with $\kappa \leqslant \mu \leqslant \tau$.

Proof. Using the definition of norms by Fourier series we have

$$\begin{aligned}
\|Lu\|_{\mu-\kappa}^2 &= \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(d,\ell)} (1+\ell)^{2(\mu-\kappa)} |(\widehat{Lu})_{\ell k}|^2 \\
&= \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(d,\ell)} (1+\ell)^{2(\mu-\kappa)} |\widehat{L}(\ell)\widehat{u}_{\ell k}|^2 \\
&\leqslant C_L^2 \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(d,\ell)} (1+\ell)^{2\mu} |\widehat{u}_{\ell k}|^2 \\
&= C_L^2 \|u\|_\mu^2
\end{aligned}$$

where we have used condition (3.2) to bound $\widehat{L}(\ell)$. On the other hand we can write

$$\begin{aligned}
\|u\|_\mu^2 &= \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(d,\ell)} (1+\ell)^{2\mu} |\widehat{u}_{\ell k}|^2 \\
&\leqslant \frac{1}{c_L^2} \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(d,\ell)} (1+\ell)^{2(\mu-\kappa)} |\widehat{L}(\ell)\widehat{u}_{\ell k}|^2 \\
&\leqslant \frac{1}{c_L^2} \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(d,\ell)} (1+\ell)^{2(\mu-\kappa)} |(\widehat{Lu})_{\ell k}|^2 \\
&= \frac{1}{c_L^2} \|Lu\|_{\mu-\kappa}^2,
\end{aligned}$$

where we have again used condition (3.2) to bound $\widehat{L}(\ell)$. ■

Now, following [11], we introduce a new positive definite kernel

$$\Psi := L^{-1}\Phi.$$

This kernel has Fourier coefficients $\widehat{\psi}(\ell) = \widehat{\phi}(\ell)/\widehat{L}(\ell)$ and defines the inner product

$$\langle u, v\rangle_\Psi := \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(d,\ell)} \frac{\widehat{L}(\ell)}{\widehat{\phi}(\ell)} \widehat{u}_{\ell k}\widehat{v}_{\ell k}, \quad u, v \in H^{\sigma+\kappa/2}, \tag{4.2}$$

with the corresponding norm $\|u\|_\Psi^2 := \langle u, u\rangle_\Psi$. Under the Assumption 3.1, we can prove that

$$c\|u\|_{\sigma+\kappa/2} \leqslant \|u\|_\Psi \leqslant C\|u\|_{\sigma+\kappa/2}, \tag{4.3}$$

with $c = \sqrt{c_L/C_\Phi}$ and $C = \sqrt{C_L/c_\Phi}$, which shows that $\|\cdot\|_\Psi$ norm is equivalent to the Sobolev norm $\|\cdot\|_{\sigma+\kappa/2}$, and the Sobolev space $H^{\sigma+\kappa/2}$ with the inner product (4.2) is a reproducing kernel Hilbert space with kernel Ψ, provided that $\sigma+\kappa/2 > d/2$.

The lemma below shows that u_N is the best approximation for u out of $V_{\Phi,X}$ in $\|\cdot\|_\Psi$ norm. The proof can be found in [11, Lemma 4].

Lemma 4.2 *Under the Assumption 3.1, if* $u \in H^{\sigma+\kappa/2}$ *and* $u_N \in V_{\Phi,X}$ *is the solution of the collocation method then*

$$\langle u - u_N, s\rangle_\Psi = 0 \text{ for all } s \in V_{\Phi,X},$$

and hence

$$\|u - u_N\|_\Psi \leqslant \|u\|_\Psi. \tag{4.4}$$

The norm equivalence property (4.3) together with (4.4) give the stability bound

$$\begin{aligned}\|u - u_N\|_{\sigma+\kappa/2} &\leqslant C\|u - u_N\|_\Psi \\ &\leqslant C\|u\|_\Psi \\ &\leqslant C\|u\|_{\sigma+\kappa/2}.\end{aligned} \tag{4.5}$$

The order of convergence of kernel methods are mainly based on the density and the quality of trial and test points. There are three geometrical quantities associated with X. The *separation distance* q_X is the radius of the largest ball that can be placed around every point in X such that no two balls overlap, i.e.

$$q_X := \frac{1}{2}\min_{j\neq k}\operatorname{dist}(x_j, x_k).$$

On the other hand, the *fill distance* corresponds to the radius of the largest empty possible ball that can be placed between the points in X. It is mathematically defined by

$$h_X := \max_{x\in\mathbb{S}^d}\min_{x_j\in X}\operatorname{dist}(x, x_j).$$

Finally the *mesh ratio* r_X is defined by

$$r_X := \frac{h_X}{q_X},$$

which measures how uniformly the points are placed. When it is close to 1, the distribution of the points in X is said to be *quasi-uniform*. For $R \geqslant 1$, let $\mathcal{X}_R = \mathcal{X}_R(\mathbb{S}^d)$ be the family of all sets of centers X with $r_X \leqslant R$; we will say that the family $\mathcal{X}_R$ is *R-uniform*.

The following "sampling inequality" or "zeros lemma" has been proved in [16] by applying Theorem 5.5 of [18].

Lemma 4.3 *Let* $\alpha, \beta \in \mathbb{R}$ *satisfy* $\beta > d/2$ *and* $0 \leqslant \alpha \leqslant \beta$. *Suppose that* $X \subset \mathbb{S}^d$ *is a set of scattered points with fill distance* h_X. *If* $u \in H^\beta$ *satisfies* $u|_X = 0$, *then for* h_X *sufficiently small, we have*

$$\|u\|_\alpha \leqslant C h_X^{\beta-\alpha}\|u\|_\beta.$$

Let us prove the first theorem which is applicable only for functions u in $H^{\sigma+\kappa/2}$. In the sequel we generalize this result to a wide range of Sobolev spaces.

Theorem 4.4 *Under the Assumption 3.1, if* $\tau := \sigma + \kappa/2$ *and* $u_N \in V_{\Phi,X}$ *is the solution of the collocation method then for all* $u \in H^\tau$ *the error bound*

$$\|u - u_N\|_\beta \leqslant C h_X^{\tau-\beta} \|u\|_\tau, \tag{4.6}$$

holds for $\kappa \leqslant \beta \leqslant \tau$ *and for sufficiently small fill distance* h_X *of set* X *on* $\mathbb{S}^d$.

Proof. By applying Theorem 4.1, Lemma 4.3 (by replacing u by $Lu - Lu_N$) and inequality (4.5) we have for $\kappa \leqslant \beta \leqslant \tau$

$$\begin{aligned} \|u - u_N\|_\beta &\leqslant C\|Lu - Lu_N\|_{\beta-\kappa} \\ &\leqslant C h_X^{\tau-\beta} \|Lu - Lu_N\|_{\tau-\kappa} \\ &\leqslant C h_X^{\tau-\beta} \|u - u_N\|_\tau \\ &\leqslant C h_X^{\tau-\beta} \|u\|_\tau, \end{aligned}$$

which complete the proof. ∎

Now, we aim to estimate the error for approximating functions smoother than those in the native space H^τ of kernel Ψ. A "doubling trick" will be applied in the case where $u \in H^{\tau+\alpha}$ for real numbers $\alpha \in [0, \tau]$. First, we measure the error in the $\|\cdot\|_\tau$ norm and then we extend it to the $\|\cdot\|_\beta$ norm for real numbers $\beta \in [\kappa, \tau]$.

Lemma 4.5 *Under the Assumption 3.1, if* $\tau := \sigma + \kappa/2$ *and* $\alpha \in [0, \tau - \kappa]$ *then*

$$\|u - u_N\|_\tau \leqslant C h_X^\alpha \|u\|_{\tau+\alpha},$$

provided that h_X *is sufficiently small and* $u \in H^{\tau+\alpha}$.

Proof. By Lemma 4.2 we have $\langle u - u_N, s\rangle_\Psi = 0$ for all $s \in V_X$. Consequently, $\langle u - u_N, u_N\rangle_\Psi = 0$ which implies $\|u - u_N\|_\Psi^2 = \langle u - u_N, u\rangle_\Psi$. Let $e := u - u_N$. We have

$$\begin{aligned}
\|u-u_N\|_\tau^2 &\leqslant \frac{C_\Phi}{c_L}\|u-u_N\|_\Psi^2 \\
&= \frac{C_\Phi}{c_L}\langle e,u\rangle_\Psi = \frac{C_\Phi}{c_L}\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(d,\ell)}\frac{\widehat{L}(\ell)}{\widehat{\phi}(\ell)}\widehat{u}_{\ell k}\overline{\widehat{e}_{\ell k}} \\
&\leqslant \frac{C_\Phi C_L}{c_\Phi c_L}\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(d,\ell)}(1+\ell)^{2\tau}\widehat{u}_{\ell k}\overline{\widehat{e}_{\ell k}} \\
&\leqslant \frac{C_\Phi C_L}{c_\Phi c_L}\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(d,\ell)}(1+\ell)^{\tau+\alpha}|\widehat{u}_{\ell k}|(1+\ell)^{\tau-\alpha}|\widehat{e}_{\ell k}| \\
&\leqslant \frac{C_\Phi C_L}{c_\Phi c_L}\left(\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(d,\ell)}(1+\ell)^{2(\tau+\alpha)}|\widehat{u}_{\ell k}|^2\right)^{1/2}\left(\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(d,\ell)}(1+\ell)^{2(\tau-\alpha)}|\widehat{e}_{\ell k}|^2\right)^{1/2} \\
&= \frac{C_\Phi C_L}{c_\Phi c_L}\|u\|_{\tau+\alpha}\|e\|_{\tau-\alpha} \\
&= \frac{C_\Phi C_L}{c_\Phi c_L}\|u\|_{\tau+\alpha}\|u-u_N\|_{\tau-\alpha},
\end{aligned}$$

where (4.3), (3.2) and (2.3) are applied in the first and third lines. On the other hand we can write

$$\begin{aligned}
\|u-u_N\|_{\tau-\alpha} &\leqslant C\|Lu-Lu_N\|_{\tau-\alpha-\kappa} && \text{(using Theorem 4.1 since } \alpha\leqslant\tau-\kappa) \\
&\leqslant Ch_X^\alpha\|Lu-Lu_N\|_{\tau-\kappa} && \text{(using Lemma 4.3 since } \tau>d/2+\kappa) \\
&\leqslant Ch_X^\alpha\|u-u_N\|_\tau. && \text{(using Theorem 4.1)}
\end{aligned}$$

Combining the two recent inequalities and then dividing both sides by $\|u-u_N\|_\tau$ yield the desired bound. ∎

Theorem 4.6 *Under the Assumption 3.1, if $\tau := \sigma+\kappa/2$, $\beta\in[\kappa,\tau]$ and $\gamma\in[\tau,2\sigma]$ then*

$$\|u-u_N\|_\beta \leqslant Ch_X^{\gamma-\beta}\|u\|_\gamma, \tag{4.7}$$

provided that h_X is sufficiently small and $u\in H^\gamma$.

Proof. By applying Theorem 4.1 and Lemma 4.3 we have

$$\begin{aligned}
\|u-u_N\|_\beta &\leqslant C\|Lu-Lu_N\|_{\beta-\kappa} && \text{(using Theorem 4.1 since } \beta\geqslant\kappa) \\
&\leqslant Ch_X^{\tau-\beta}\|Lu-Lu_N\|_{\tau-\kappa} && \text{(using Lemma 4.3 since } \tau\geqslant d/2+\kappa) \\
&\leqslant Ch_X^{\tau-\beta}\|u-u_N\|_\tau. && \text{(using Theorem 4.1)}
\end{aligned}$$

An application of Lemma 4.5 for $\alpha=\gamma-\tau$ then gives the desired bound. ∎

We note that, the same error bound has been proved in [21, Theorem 6.3] using a different approach which uses well-known knowledge on the Galerkin method. The analysis is based on an observation that the collocation method can be viewed as a Galerkin method, due to the reproducing kernel property of the space in use. Our proof above is different and in addition the following theorem concerns the case $\gamma \leqslant \tau$, i.e. the case where u lies outside the native space H^τ of kernel Ψ.

Theorem 4.7 *Under the Assumption 3.1, let $\tau := \sigma + \kappa/2$, $\gamma > d/2 + \kappa$ and $\kappa \leqslant \beta \leqslant \gamma \leqslant \tau$. Then for all $u \in H^\gamma$ we have*

$$\|u - u_N\|_\beta \leqslant C r_X^{\tau-\gamma} h_X^{\gamma-\beta} \|u\|_\gamma, \tag{4.8}$$

provided that h_X is sufficiently small. Here r_X is the mesh ratio of set X.

Proof. Remember that Lu_N can be viewed as the Λ-interpolant of function Lu where $\Lambda = L\Phi$. Inspiring from (3.5), the inner product on $\mathcal{N}_\Lambda$ is defined by

$$\langle u, v\rangle_\Lambda := \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} \frac{1}{\widehat{L}(\ell)\widehat{\phi}(\ell)} \widehat{u}_{\ell k} \widehat{v}_{\ell k},$$

with the corresponding norm $\|u\|_\Lambda^2 := \langle u, u\rangle_\Lambda$. Under the Assumption 3.1, we can prove that

$$c\|u\|_{\sigma-\kappa/2} \leqslant \|u\|_\Lambda \leqslant C\|u\|_{\sigma-\kappa/2},$$

which shows that if $\sigma - \kappa/2 > d/2$ then $H^{\sigma-\kappa/2}$ is Λ reproducing kernel Hilbert space and norms $\|\cdot\|_\Lambda$ and $\|\cdot\|_{\sigma-\kappa/2}$ are equivalent.

By assumptions $\gamma > d/2 + \kappa$ and $\kappa \leqslant \beta \leqslant \gamma \leqslant \tau$. Let $\bar{\gamma} = \gamma - \kappa$, $\bar{\beta} = \beta - \kappa$ and $\bar{\tau} = \tau - \kappa$. Since $d/2 < \bar{\gamma} \leqslant \bar{\tau}$, [18, Theorem 5.5] yields

$$\|Lu - Lu_N\|_{\bar{\beta}} \leqslant C r_X^{\bar{\tau}-\bar{\gamma}} h_X^{\bar{\gamma}-\bar{\beta}} \|Lu\|_{\bar{\gamma}} = C r_X^{\tau-\gamma} h_X^{\gamma-\beta} \|Lu\|_{\bar{\gamma}},$$

because $\bar{\gamma} - \bar{\beta} = \gamma - \beta$ and $\bar{\tau} - \bar{\gamma} = \tau - \gamma$. Now, by applying Theorem 4.1 and the above inequality we have

$$\begin{aligned} \|u - u_N\|_\beta &\leqslant C\|Lu - Lu_N\|_{\bar{\beta}} \\ &\leqslant C r_X^{\tau-\gamma} h_X^{\gamma-\beta} \|Lu\|_{\bar{\gamma}} \\ &\leqslant C r_X^{\tau-\gamma} h_X^{\gamma-\beta} \|u\|_\gamma, \end{aligned}$$

which proves (4.8). ■

If we work with R-uniform family of centers, i.e. $X \in \mathcal{X}_R$, then in inequality (4.8) we can replace r_X by R to get the bound

$$\|u - u_N\|_\beta \leqslant C_{\mathcal{X}_R} h_X^{\gamma-\beta} \|u\|_\gamma, \tag{4.9}$$

where $Cr_X^{\tau-\gamma} \leqslant CR^{\tau-\gamma} =: C_{\mathcal{X}_R}$.

An L_2-error bound follows trivially. From Theorem 4.1 we have $\|Lu - Lu_N\|_0 \leqslant C\|u - u_N\|_\kappa$. Combining this with the special case $\beta = \kappa$ in (4.7) and (4.9) for $X \in \mathcal{X}_R$, we simply have

$$\|Lu - Lu_N\|_0 \leqslant C\, h_X^{\gamma-\kappa}\|u\|_\gamma, \quad d/2 + \kappa < \gamma \leqslant 2\sigma.$$

On the other hand we have $\|u - u_N\|_0 \leqslant C\|Lu - Lu_N\|_0$. Thus the following L_2-error bound can be deduced,

$$\|u - u_N\|_0 \leqslant C\, h_X^{\gamma-\kappa}\|u\|_\gamma, \quad d/2 + \kappa < \gamma \leqslant 2\sigma, \tag{4.10}$$

for all $u \in H^\gamma$ satisfying (3.1). Although $\|u - u_N\|_0 \leqslant C\|Lu - Lu_N\|_0$ is a coarse estimate, the same example as that is given in [11, Lemma 6] shows that (in general) the error bound (4.10) is sharp not only for $\gamma = \sigma + \kappa/2$ (the case in [11]) but also for all $\gamma \in (d/2 + \kappa, 2\sigma]$.

Here we discuss about the generalized Hermite collocation method for PDE problem (3.1) where the approximation space

$$W_{\Phi,X} := \operatorname{span}\{L\Phi(\cdot, x_j),\ x_j \in X\}$$

is used instead of $V_{\Phi,X}$ and the numerical solution

$$u_N = u_{N,X,\Phi} = \sum_{j=1}^{N} b_j L\Phi(\cdot, x_j)$$

is obtained by enforcing the collocation conditions (3.3). This leads to linear system $Ab = F$ where $a_{kj} = LL\Phi(x_k, x_j),\ k, j = 1, \dots, N$ and $f_k = f(x_k),\ k = 1, \dots, N$. In the case of the sphere (since there exists no boundary) this method is equivalent to the previous one by replacing Φ by $L\Phi$. The auxiliary kernel Ψ will be $\Psi = L^{-1}(L\Phi) = \Phi$. Since the Fourier coefficients of $L\Phi$ behave like as $(1 + \ell)^{-2(\sigma-\kappa/2)}$ rather than $(1 + \ell)^{-2\sigma}$ for Φ itself, we just need to replace σ in the above analysis by $\sigma - \kappa/2$ to end with the following corollaries.

Corollary 4.8 *Assume that $u_N \in W_{\Phi,X}$ is the collocation solution for* (3.1) *where L satisfies* (3.2) *and Φ satisfies* (2.3) *for $\sigma > \kappa + d/2$. Let $\beta \in [\kappa, \sigma]$ and $\gamma \in [\sigma, 2\sigma - \kappa]$. Then the error bound*

$$\|u - u_N\|_\beta \leqslant C h_X^{\gamma-\beta}\|u\|_\gamma,$$

holds provided that h_X is sufficiently small and $u \in H^\gamma$.

Corollary 4.9 *Assume that $u_N \in W_{\Phi,X}$ is the collocation solution for* (3.1) *where L satisfies* (3.2) *and Φ satisfies* (2.3) *for $\sigma > \kappa + d/2$. If $\gamma > d/2 + \kappa$ and $\kappa \leqslant \beta \leqslant \gamma \leqslant \sigma$ then for all $u \in H^\gamma$ we have*

$$\|u - u_N\|_\beta \leqslant C r_X^{\sigma-\gamma} h_X^{\gamma-\beta} \|u\|_\gamma,$$

provided that h_X is sufficiently small.

Also, the L_2-error bound (4.10) can be rewritten similarly for collocation solution $u_N \in W_{\Phi,X}$. The only difference is that the rage of γ should now be changed to $d/2 + \kappa < \gamma \leqslant 2\sigma - \kappa$.

We remark that the Sobolev norm $\|u\|_\gamma$ on the right hand sides of the all above error bounds can be replaced by $\|f\|_{\gamma-\kappa}$ by applying the norm equivalence property (4.1). This makes the measurements more accessible because f is a known function while u is the unknown solution.

We close this section by noting that if n is a nonnegative integer such that $\beta > n + d/2$, then using the Sobolev Imbedding Theorem we can rewrite error bounds (4.6), (4.7), (4.8) and (4.9) with the error measured in the norm of $C^n(\mathbb{S}^d)$ rather than in the Sobolev norm $\|\cdot\|_\beta$. In particular, if $n = 0$, then we can bound the pointwise error.

5 Numerical experiments

Consider equation (3.1) on $\mathbb{S}^2$ where $L = -\Delta_0 + I$. In order to test and verify the theoretical error bounds of preceding section we need to construct a finitely smooth true solution u for (3.1). Let $\{\xi_1, \dots, \xi_M\}$ be a set of M points on $\mathbb{S}^2$ and define

$$u(x) := \sum_{k=1}^{M} b_k \varphi_\alpha(\sqrt{2 - 2x^T \xi_k}), \quad x \in \mathbb{S}^2,$$

for some known coefficients b_k, where

$$\varphi_\alpha(r) = (\varepsilon r)^{\alpha-3/2} K_{\alpha-3/2}(\varepsilon r)$$

is the well-known Matérn kernel. Here K_α is the modified Bessel function of the second kind of order α. Let $\alpha = (\mu+2)/2$. Since φ_α produces $H^\alpha(\mathbb{R}^3)$, its restriction to $\mathbb{S}^2$ produces $H^{\alpha-1/2}(\mathbb{S}^2)$. Then [13, Lemma 8.3] can be applied to show $u \in H^\gamma(\mathbb{S}^2)$ for any $\gamma < \mu$. We write $u \in H^{\mu-\epsilon}$ where $\epsilon > 0$ is arbitrary small real number. We use various α's to verify the error bounds (4.6), (4.7) and (4.9). In experiment the shape parameter $\varepsilon = 2$ and a set $\{\xi_1, \dots, \xi_{100}\}$ of scattered points on $\mathbb{S}^2$ [22] are used. Moreover, we set

$$\tilde{b} = (0.1, -0.2, 0.4, 0.3, -0.1, -0.4, 0.3, -0.5, 0.1, 0.2), \quad b = (\underbrace{\tilde{b}, -\tilde{b}, \tilde{b}, \dots, -\tilde{b}}_{10 \text{ times}}).$$

The right hand side function f is calculated, accordingly. The Wendland's kernel

$$\Phi(x) = (1 - \|x\|_2)_+^6 (35\|x\|_2^2 + 18\|x\|_2 + 3), \quad x \in \mathbb{R}^3,$$

restricted to $\mathbb{S}^2$ is employed to form the trial space. This kernel satisfies (2.3) with $\sigma = 3.5$. Since $\tau = \sigma + \kappa/2 = 4.5$, the constructed solution u is less smoother than H^τ functions (u is outside the native space of $\Psi = L^{-1}\Phi$) if $\mu \leqslant 4.5$ or $\alpha \leqslant 3.25$. Otherwise, u is smoother than the functions in H^τ. Since $u \in H^{\mu-\epsilon}$ the error bounds (4.6), (4.7) and (4.9) predict the order $\mu - \epsilon - \beta$ if the error function $e_N := u - u_N$ is measured in $\|\cdot\|_\beta$ norm. We assume $\beta = 2$ and approximate the error $\|e_N\|_2$ by $\|e_N + \Delta_0 e_N\|_0$ where the L_2 error is computed via a spherical quadrature on $\mathbb{S}^2$ adopted from [6]. The equal area partitioning algorithm [22] is used to generate the sets of scattered quasi-uniform points on $\mathbb{S}^2$ that are used as trial and test (collocation) points. Results are given in Table 1. Since for X in a family of R-uniform sets on $\mathbb{S}^2$ we have $h_X = \mathcal{O}(N^{-1/2})$, the numerical orders are computed via

$$\log\left(\frac{\|e_{\text{old}}\|_2}{\|e_{\text{new}}\|_2}\right) \Big/ \log\left(\sqrt{\frac{N_{\text{new}}}{N_{\text{old}}}}\right),$$

row by row. The theoretical orders are shown in the last row of Table 1. As we see, the numerical orders confirm the theoretical ones, approximately. In the first case ($\alpha = 3$) the true solution is outside the native space of Ψ (or $H^{4.5}$), thus the error bound (4.9) is applicable. The last column shows that the order of convergence is saturated at full order $2\sigma - \beta = 7 - 2 = 5$ even if u is smoother than H^7.

6 Conclusion

This paper concerns the error analysis of the kernel collocation methods for partial differential equations on the unit sphere $\mathbb{S}^d$. Restricted positive definite kernels from $\mathbb{R}^{d+1}$ into $\mathbb{S}^d$ are used and the analysis is given for functions in wider range of Sobolev spaces. The experimental results support the theoretical bounds.

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	$\alpha = 3\,(u \in H^{4-\epsilon})$		$\alpha = 3.5\,(u \in H^{5-\epsilon})$		$\alpha = 4.5\,(u \in H^{7-\epsilon})$		$\alpha = 5.5\,(u \in H^{9-\epsilon})$	
N	$\|e_N\|_2$	orders	$\|e_N\|_2$	orders	$\|e_N\|_2$	orders	$\|e_N\|_2$	orders
40	8.18e − 1	–	8.48e − 1	–	9.08e − 1	–	9.48e − 1	–
80	4.46e − 1	1.75	4.41e − 1	1.89	5.04e − 1	1.70	5.47e − 1	1.58
160	1.65e − 1	2.87	1.17e − 1	3.84	1.41e − 1	3.68	1.59e − 1	3.57
320	7.04e − 2	2.46	2.44e − 2	4.51	2.74e − 2	4.72	3.12e − 2	4.69
640	3.76e − 2	1.81	6.13e − 3	3.99	4.87e − 3	4.98	5.57e − 3	4.97
1280	1.84e − 2	2.05	1.73e − 3	3.64	8.65e − 4	4.99	9.90e − 4	4.98
2560	9.54e − 3	1.90	5.92e − 4	3.10	1.49e − 4	5.08	1.70e − 4	5.08
5120	4.79e − 3	1.99	1.97e − 4	3.18	2.63e − 5	5.01	3.01e − 5	5.01
Theoretical order		$2-\epsilon$		$3-\epsilon$		$5-\epsilon$		5

Table 1: The relative H^2-norm of $e_N = u - u_N$ for various smooth solutions u together with the numerical and theoretical orders.

References

[1] F. Dai and Y. Xu. *Approximation Theory and Harmonic Analysis on Spheres and Balls*. Springer, New York, 2013.

[2] C. Franke and R. Schaback. Convergence order estimeates of meshless collocation methods using radial basis functions. *Adv. Comput. Math.*, 8:381–399, 1998.

[3] C. Franke and R. Schaback. Solving partial differential equations by collocation using radial basis functions. *Appl. Math. Comput.*, 93:73–82, 1998.

[4] E. Fuselier and G. B. Wright. Scattered data interpolation on embeded submanifolds with restricted positive definite kernels: Sobolev error estimates. *SIAM J. Numer. Anal.*, 50:1753–1776, 2012.

[5] P. Giesl and H. Wendland. Meshless collocation: error estimates with application to dynamical systems. *SIAM J. Numer. Anal.*, 45:1723–1741, 2007.

[6] K. Hesse and R. S. Womersley. Numerical integration with polynomial exactness over a spherical cap. *Adv. Comput. Math.*, 36:451–483, 2012.

[7] Y. C. Hon and R. Schaback. On unsymmetric collocation by radial basis functions. *Appl. Math. Comput.*, 119:177–186, 2001.

[8] E. J. Kansa. Multiquadrics —A scattered data approximation scheme with applications to computational fluid-dynamics, I. Surface approximations and partial derivative estimates. *Comput. Math. Appl.*, 19:127–145, 1990.

[9] E. J. Kansa. Multiquadrics —A scattered data approximation scheme with applications to computational fluid-dynamics, II. Solutions to parabolic, hyperbolic and elliptic partial differential equations. *Comput. Math. Appl.*, 19:147–161, 1990.

[10] Q. T. Le Gia. Galerkin approximation for elliptic PDEs on spheres. *J. Approx. Theory*, 130:123–147, 2004.

[11] Q. T. Le Gia, I. H. Sloan, and H. Wendland. Multiscale RBF collocation for solving PDEs on spheres. *Numer. Math.*, 121:99–125, 2012.

[12] J. Levesley and Z. Luo. Error estimates for Hermite interpolation on spheres. *J. Math. Anal. Appl.*, 281:46–61, 2003.

[13] D. Mirzaei. A Petrov–Galerkin kernel approximation on the sphere. *SIAM J. Numer. Anal.*, 56:274–295, 2018.

[14] T. M. Morton and M. Neamtu. Error bonds for solving pseudodifferential equations on spheres. *J. Approx. Theory*, 114:242–268, 2002.

[15] C. Müler. *Spherical Harmonics, Lecture Notes in Mathematics.* Springer-Varlag, Berlin, 1966.

[16] F. J. Narcowich, S. T. Rowe, and J. D. Ward. A novel Galerkin method for solving PDEs on the sphere using highly localized kernel bases. *Math. Comput.*, 86:197–231, 2017.

[17] F. J. Narcowich, X. Sun, and J. D. Ward. Approximation power of RBFs and their associated SBFs: A connection. *Adv. Comput. Math.*, 27:107–124, 2007.

[18] F. J. Narcowich, X. Sun, J. D. Ward, and H. Wendland. Direct and inverse Sobolev error estimates for scattered data interpolation via spherical basis functions. *Found. Comput. Math.*, 7:369–390, 2007.

[19] F. J. Narcowich and J. D. Ward. Generalized Hermite interpolation via matrix-valued conditionally positive definite functions. *Math. Comput.*, 63:661–687, 1994.

[20] F. J. Narcowich and J. D. Ward. Scattered-data interpolation on spheres: error estimates and locally supported basis functions. *SIAM J. Math. Anal.*, 33:1393–1410, 2002.

[21] T. D. Pham and T. Tran. Strongly elliptic pseudodifferential equations on the sphere with radial basis functions. *Numer. Math.*, 128:589–614, 2014.

[22] E. B. Saff and A. B. J. Kuijlaars. Distributing many points on a sphere. *Math. Intell.*, 19:5–11, 1997.

[23] R. Schaback. Convergence of unsymmetric kernel-based meshless collocation methods. *SIAM J. Numer. Anal.*, 45:333–351, 2007.

[24] R. Schaback. Unsymmetric meshless methods for operator equations. *Numer. Math.*, 114:629–651, 2010.

[25] I. J. Schoenberg. Positive definite function on spheres. *Duke Math. J.*, 9:96–108, 1942.

[26] H. Wendland. *Scattered Data Approximation.* Cambridge University Press, 2005.

[27] H. Wendland. On the stability of meshless symmetric collocation for boundary value problems. *BIT Numer. Math.*, 47:455–468, 2007.

[28] Z. Wu. Hermite Birkhoff interpolation of scattered data by radial basis functions. *Approx. Theory Appl.*, 8:1–10, 1992.

[29] Y. Xu and E. W. Cheney. Strictly positive definite functions on spheres. *Proc. Am. Math. Soc.*, 116:977–981, 1992.

[30] W. zu Castell and F. Filbir. Radial basis functions and corresponding zonal series expansions on the sphere. *J. Approx. Theory*, 134:65–79, 2005.