Spline Spaces

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1 Introduction

Spline functions form one of the central tools of approximation theory and computer-aided geometric design. A spline is a piecewise polynomial function with prescribed smoothness at the joining points, known as *knots*. By adjusting the degree of the polynomials and the degree of smoothness across the knots, one can construct spaces suitable for interpolation, approximation, or geometric modeling.

In this note, we study spline spaces of polynomial degree ℓ with varying smoothness levels, ranging from C^0 up to $C^{\ell-1}$, which have received less attention in the literature.

2 Definition and examples

We denote by

$$X = \{x_0, x_1, \dots, x_n\}, \qquad x_0 < x_1 < \cdots < x_n,$$

a partition of the interval $[x_0, x_n]$ into subintervals $[x_{i-1}, x_i]$.

Definition 2.1. Let $\ell \geq 0$ be an integer denoting the polynomial degree, and let k be the smoothness parameter with $0 \leq k \leq \ell - 1$. We define the space of *splines of degree ℓ and smoothness C^k* on the knot set X by

$$\mathbb{S}_X^{\ell,k} = \left\{ s \in C^k[x_0, x_n] : \ s|_{[x_{i-1},x_i]} \in \mathbb{P}_\ell, \ i = 1, \dots, n \right\},$$

where $\mathbb{P}_\ell$ denotes the space of all polynomials of degree at most ℓ.

In other words, $\mathbb{S}_X^{\ell,k}$ consists of all functions that are piecewise polynomials of degree ℓ, with continuous derivatives up to order k at each interior knot.

Each subinterval $[x_{i-1}, x_i]$ contributes $(\ell + 1)$ polynomial coefficients, giving $(n)(\ell + 1)$ total coefficients before enforcing continuity. At each of the $(n - 1)$ interior knots, $k + 1$ continuity conditions are imposed. Hence,

$$\dim \mathbb{S}_X^{\ell,k} = n(\ell + 1) - (n - 1)(k + 1) = (\ell - k)n + k + 1.$$

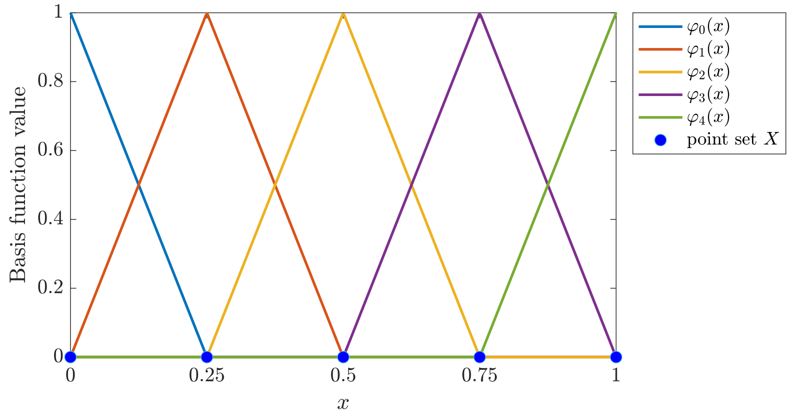

Figure 1: Hat functions form a basis for spline space $\mathbb{S}_X^{1,0}$ (The C^0 linear spline space). Here $X = \{0,\, 0.25,\, 0.5,\, 0.75,\, 1\}$.

Example 2.2 (Piecewise linear splines ($\ell = 1$, $k = 0$))**.** In this case, each segment of the spline is a straight line, and only function continuity is enforced at the knots. The corresponding spline space $\mathbb{S}_X^{1,0}$ consists of continuous, piecewise linear functions defined on the partition X. The dimension of this space is $\dim \mathbb{S}_X^{1,0} = n + 1$, and a convenient basis is given by the standard *hat functions* (or nodal basis functions)

$$\varphi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}}, & x_{j-1} \le x < x_j, \\ \dfrac{x_{j+1} - x}{x_{j+1} - x_j}, & x_j \le x < x_{j+1}, \\ 0, & \text{otherwise}, \end{cases} \qquad j = 0, 1, \dots, n.$$

To construct the full basis, we first extend the knot set to

$$X_{\text{ext}} = \{x_{-1}, x_0, x_1, \dots, x_n, x_{n+1}\},$$

and then restrict the resulting functions to the physical interval $[x_0, x_n]$. Consequently, the first and last basis functions, φ_0 and φ_n, are one-sided.

The basis $\{\varphi_j\}_{j=0}^n$ is *cardinal*, meaning that $\varphi_j(x_i) = \delta_{ij}$. Consequently, any function f defined on the set X admits an interpolant from the space $\mathbb{S}_X^{1,0}$ of the form

$$s(x) = \sum_{j=0}^{n} \varphi_j(x)\, f(x_j).$$

This space coincides with the function space used in the linear finite element method (FEM).

Figure 1 illustrates the hat functions for the knot sequence $X = \{0,\, 0.25,\, 0.5,\, 0.75,\, 1\}$. Each φ_i is nonnegative, locally supported, and the set $\{\varphi_j\}_{j=0}^n$ forms a partition of unity:

$$\sum_{j=0}^{n} \varphi_j(x) = 1, \qquad \forall\, x \in [x_0, x_n].$$

Example 2.3 (Cubic splines)**.** For $\ell = 3$, two levels of smoothness are of special interest:

(i) C^2 cubic splines ($k = 2$): These are the standard cubic splines widely used for interpolation and smoothing. They are C^2 across the knots and form the space $\mathbb{S}_X^{3,2}$. The dimension of this space is $\dim \mathbb{S}_X^{3,2} = n + 3$, which exceeds the number of interpolation points ($n + 1$ points) by 2. Consequently, the interpolation problem on the set X is not uniquely determined and requires two additional conditions, typically imposed at the endpoints x_0 and x_n, for example, by prescribing the derivative values at x_0 and x_n.

(ii) C^1 cubic splines ($k = 1$): These are more general; they have C^1 smoothness but may have discontinuous second derivatives at the knots. The dimension of this space is $\dim \mathbb{S}_X^{3,1} = 2n + 2$, which is approximately twice that of the C^2 cubic spline space. This increase in dimension is natural – as the smoothness requirement decreases, the corresponding approximation space becomes larger. However, an interpolation problem in this space requires $2n+2$ interpolation conditions, that is, twice as many as the number of knots in X. When the additional interpolation conditions are imposed on the first derivatives at *all* the points in X, the resulting interpolant is the *cubic Hermite interpolation*. Alternatively, one may employ the same spline space in a standard Lagrange-type interpolation scheme using $2n + 2$ prescribed function values. These C^1 cubic splines are also widely used in finite element formulations, particularly those based on Hermite-type basis functions.

Finding a basis for the general spline space $\mathbb{S}_X^{\ell,k}$ is not as straightforward as in the case of linear splines. The next section is devoted to addressing this important problem.

3 Bases for spline spaces

Different bases can be used to represent elements of $\mathbb{S}_X^{\ell,k}$, depending on the application and desired properties.

3.1 One-sided (truncated power) functions

Let $(x - t)_+^m$ denote the truncated power function

$$(x - t)_+^m := \begin{cases} (x - t)^m, & x \geq t, \\ 0, & x < t, \end{cases} \qquad m \geq 0.$$

It is obvious that $(x - t)_+^m$ has only $m - 1$ times continuous derivatives at point $x = t$, thus $(\cdot - t)_+^m \in C^{m-1}(\mathbb{R})$. For a general C^k spline space $\mathbb{S}_X^{\ell,k}$, a basis can be constructed from a polynomial part $p(x)$ of degree ℓ (or equivalently, the monomials $1, x, \ldots, x^\ell$), and truncated power functions of the form $(x - x_j)_+^i$ with

$$k + 1 \leq i \leq \ell, \quad j = 1, \ldots, n - 1,$$

where x_j are the interior knots.

Theorem 3.1 (Polynomials plus truncated power basis for $\mathbb{S}_X^{\ell,k}$)**.** Let $X = \{x_0 < x_1 < \cdots < x_n\}$, $\ell \geq 1$, and $0 \leq k \leq \ell - 1$. Then the set of functions

$$\{1, x, \ldots, x^\ell\} \ \cup \ \{(x - x_j)_+^i : j = 1, \ldots, n-1, \ k+1 \leq i \leq \ell\}$$

forms a basis for the spline space $\mathbb{S}_X^{\ell,k}$ of piecewise polynomials of degree at most ℓ with C^k continuity at the knots.

Proof. We first verify that the number of functions equals the dimension of $\mathbb{S}_X^{\ell,k}$. The polynomial part contributes $\ell + 1$ functions. For each interior knot x_j, there are $\ell - k$ truncated power functions $(x - x_j)_+^i$ with $i = k+1, \ldots, \ell$, giving a total of $(n-1)(\ell - k)$ functions. Hence the total number of functions is

$$(\ell + 1) + (n-1)(\ell - k) = (\ell - k)n + k + 1,$$

which equals $\dim \mathbb{S}_X^{\ell,k}$.

Next, we show that these functions indeed span the space $\mathbb{S}_X^{\ell,k}$. Let $s \in \mathbb{S}_X^{\ell,k}$. We claim that s admits the representation

$$s(x) = \sum_{i=0}^{\ell} c_i x^i + \sum_{j=1}^{n-1} \sum_{i=k+1}^{\ell} a_{i,j} (x - x_j)_+^i, \quad \text{for all } x \in [x_0, x_n]. \tag{3.1}$$

We establish this form by induction over the subintervals of X. On the first interval $I_1 = [x_0, x_1]$, the function s is a polynomial of degree ℓ, and hence

$$s(x) = c_0 + c_1 x + \cdots + c_\ell x^\ell, \qquad x \in I_1.$$

This shows that the representation

$$s(x) = \sum_{i=0}^{\ell} c_i x^i + \sum_{j=1}^{m-1} \sum_{i=k+1}^{\ell} a_{i,j} (x - x_j)_+^i, \quad x \in I_m = [x_0, x_m], \tag{3.2}$$

holds for $m = 1$, since the empty sum $\sum_{j=1}^{0}$ is interpreted as zero. Now, assume that (3.2) is valid for some $m \geq 1$. We define

$$y(x) := s(x) - \sum_{i=0}^{\ell} c_i x^i - \sum_{j=1}^{m-1} \sum_{i=k+1}^{\ell} a_{i,j} (x - x_j)_+^i, \quad x \in I_{m+1} = [x_0, x_{m+1}]. \tag{3.3}$$

By construction, $y \in C^k(I_{m+1})$, and we have $y(x) = 0$ for $x \in I_m$, while y coincides with a polynomial of degree ℓ on $[x_m, x_{m+1}]$. Therefore, y satisfies the initial value problem

$$\begin{cases} y^{(\ell+1)}(x) = 0, & x \geq x_m, \\ y^{(i)}(x_m) = 0, & i = 0, 1, \ldots, k. \end{cases}$$

The general solution to this system is

$$y(x) = \sum_{i=k+1}^{\ell} a_{i,m} (x - x_m)^i, \qquad x \geq x_m,$$

for some constants $a_{i,m}$. Since $y(x) = 0$ for $x \le x_m$, it follows that

$$y(x) = \sum_{i=k+1}^{\ell} a_{i,m}(x - x_m)_+^i, \qquad x \in I_{m+1}.$$

Substituting this expression into (3.3) establishes the desired representation on $I_{m+1} = [x_0, x_{m+1}]$. Continuing inductively over all subintervals yields the global representation (3.1) on $[x_0, x_n]$.

Finally, we establish the linear independence of the functions in (3.1). Assume that $s(x) \equiv 0$, where s admits the representation (3.1). On the first interval $[x_0, x_1]$, all truncated power terms vanish, and hence

$$c_0 + c_1 x + \cdots + c_\ell x^\ell = 0, \qquad x \in [x_0, x_1].$$

Since the monomials $\{1, x, \ldots, x^\ell\}$ are linearly independent, it follows that $c_i = 0$ for all $i = 0, \ldots, \ell$. Next, consider the interval $[x_1, x_2]$. Here, the only active truncated power terms correspond to the knot x_1, so we have

$$\sum_{i=k+1}^{\ell} a_{i,1}(x - x_1)_+^i = \sum_{i=k+1}^{\ell} a_{i,1}(x - x_1)^i = 0, \qquad x \in [x_1, x_2].$$

The polynomials $\{(x - x_1)^{k+1}, \ldots, (x - x_1)^\ell\}$ are linearly independent, and thus $a_{i,1} = 0$ for all $i = k+1, \ldots, \ell$. Proceeding inductively, we note that on each subsequent interval $[x_j, x_{j+1}]$, the only remaining active terms are those associated with x_j. Applying the same argument yields $a_{i,j} = 0$ for all $i = k+1, \ldots, \ell$ and $j = 1, \ldots, n-1$. Consequently, all coefficients c_i and $a_{i,j}$ vanish, and hence the set of functions in (3.1) is linearly independent. $\square$

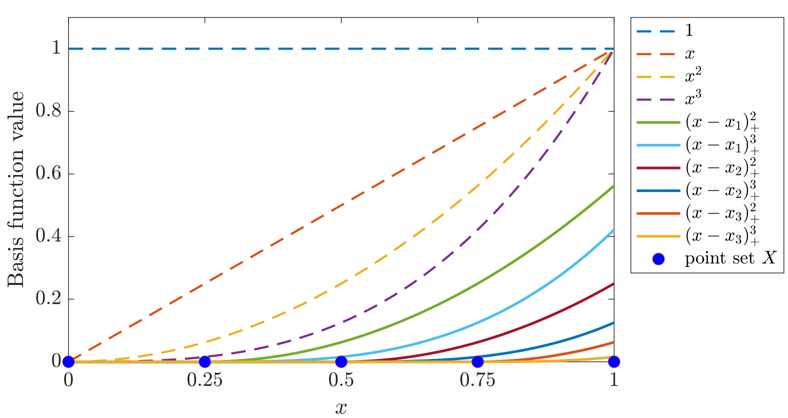

Figure 2: Polynomials plus truncated power functions form a basis for spline space $\mathbb{S}_X^{3,1}$ (the C^1 cubic spline space) where $X = \{x_0, x_1, x_2, x_3, x_4\} = \{0,\ 0.25,\ 0.5,\ 0.75,\ 1\}$.

Remark. When $k = \ell - 1$, only the highest power $(x - x_j)_+^\ell$ appears, recovering the classical Curry–Schoenberg basis for maximally smooth splines. For $k < \ell - 1$, the lower-order truncated powers represent the allowed derivative discontinuities at the interior knots.

Example 3.2. For C^1 cubic splines ($\ell = 3$, $k = 1$) on knots $x_0 < x_1 < \cdots < x_n$, the truncated power basis includes $(x - x_j)_+^2$ and $(x - x_j)_+^3$ for each interior knot x_j. The polynomial part is cubic, giving the standard dimension $\dim \mathbb{S}_X^{3,1} = 2n + 2$. Figure 2 shows the basis function for $\mathbb{S}_X^{3,1}$ where $X = \{0,\ 0.25,\ 0.5,\ 0.75,\ 1\}$. As we see, these basis functions form a non-compactly supported basis for this spline space.

3.2 Hermite basis functions (in a special case)

In the spline space $\mathbb{S}_X^{\ell,k}$, with ℓ denoting the polynomial degree and k the smoothness at interior knots, the *Hermite basis* arises naturally when $\ell = 2k + 1$. In this case, the number of available polynomial coefficients per interval exactly matches the number of continuity conditions at the endpoints, yielding a unique basis associated with function values and derivatives up to order k at each knot.

Let $X = \{x_0, x_1, \ldots, x_n\}$, with $x_0 < x_1 < \cdots < x_n$, and let $\mathbb{S}_X^{\ell,k}$ denote the space of all piecewise polynomials of degree ℓ whose derivatives up to order k are continuous on $[x_0, x_n]$. We assume that $\ell = 2k + 1$, so that each polynomial segment on $[x_{i-1}, x_i]$ has $\ell + 1 = 2(k + 1)$ degrees of freedom, corresponding exactly to the values of the function and its first k derivatives at both endpoints of the interval.

Theorem 3.3 (Hermite basis for $\ell = 2k + 1$)**.** Let $\ell = 2k + 1$. Then, for a given knot set $X = \{x_0, x_1, \ldots, x_n\}$, there exist $N = (n + 1)(k + 1)$ basis functions $\{\phi_{i,r}(x)\}$, with $i = 0, \ldots, n$ and $r = 0, \ldots, k$, such that

$$\phi_{i,r}^{(s)}(x_j) = \delta_{ij}\,\delta_{rs}, \qquad 0 \le r, s \le k, \quad 0 \le i, j \le n.$$

Moreover, every spline $s \in \mathbb{S}_X^{2k+1,k}$ can be uniquely expressed as

$$s(x) = \sum_{i=0}^{n} \sum_{r=0}^{k} s^{(r)}(x_i)\,\phi_{i,r}(x).$$

Proof. On each interval $[x_i, x_{i+1}]$, a polynomial of degree $\ell = 2k + 1$ has $\ell + 1 = 2(k + 1)$ coefficients. Imposing function and derivative matching of orders $0, 1, \ldots, k$ at both endpoints of the interval requires $2(k+1)$ conditions, which determine the polynomial uniquely. Thus, a Hermite representation exists locally on each interval. The global basis is then constructed by patching these local functions together, sharing the degrees of freedom corresponding to the common endpoints. The total number of distinct endpoint degrees of freedom is $(n + 1)(k + 1)$, equal to the dimension of $\mathbb{S}_X^{2k+1,k}$, ensuring a one-to-one correspondence between coefficient sets and basis functions. □

Local construction on each interval. For a fixed interval $[x_i, x_{i+1}]$ with length $h_i = x_{i+1} - x_i$, we introduce the normalized variable

$$t = \frac{x - x_i}{h_i}, \qquad t \in [0, 1].$$

Let $\{H_{pq}(t)\}_{p,q=0}^{k}$ denote the *local Hermite basis* on $[0,1]$, defined by

$$H_{pq}^{(r)}(0) = \delta_{pr}\,\delta_{q0}, \qquad H_{pq}^{(r)}(1) = \delta_{pr}\,\delta_{q1}, \qquad 0 \le r \le k.$$

These polynomials satisfy the endpoint interpolation conditions for function values and derivatives up to order k. The local Hermite interpolant on $[x_i, x_{i+1}]$ is then

$$s_i(x) = \sum_{r=0}^{k} \left[s^{(r)}(x_i)\, h_i^r\, H_{r0}\!\left(\tfrac{x-x_i}{h_i}\right) + s^{(r)}(x_{i+1})\, h_i^r\, H_{r1}\!\left(\tfrac{x-x_i}{h_i}\right) \right].$$

Global basis functions. The global Hermite basis $\{\phi_{i,r}(x)\}$ can now be described piecewise. For an interior knot x_i ($1 \le i \le n-1$) and derivative order r, the function $\phi_{i,r}(x)$ is supported on the two intervals adjacent to x_i, and is defined by

$$\phi_{i,r}(x) = \begin{cases} h_{i-1}^r\, H_{r1}\left(\dfrac{x-x_{i-1}}{h_{i-1}}\right), & x_{i-1} \le x \le x_i, \\[2ex] h_i^r\, H_{r0}\left(\dfrac{x-x_i}{h_i}\right), & x_i \le x \le x_{i+1}, \\[2ex] 0, & \text{otherwise.} \end{cases}$$

For the endpoints $i=0$ and $i=n$, the basis reduces to a single interval:

$$\phi_{0,r}(x) = \begin{cases} h_0^r\, H_{r0}\left(\dfrac{x-x_0}{h_0}\right), & x_0 \le x \le x_1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\phi_{n,r}(x) = \begin{cases} h_{n-1}^r\, H_{r1}\left(\dfrac{x-x_{n-1}}{h_{n-1}}\right), & x_{n-1} \le x \le x_n, \\ 0, & \text{otherwise.} \end{cases}$$

Example 3.4 (Cubic Hermite basis ($\ell = 3$, $k = 1$))**.** For the cubic case ($k = 1$), the local Hermite polynomials on $[0,1]$ are

$$\begin{aligned} H_{00}(t) &= 2t^3 - 3t^2 + 1, \quad & H_{10}(t) &= t^3 - 2t^2 + t, \\ H_{01}(t) &= -2t^3 + 3t^2, & H_{11}(t) &= t^3 - t^2. \end{aligned}$$

Then, on each interval $[x_i, x_{i+1}]$,

$$s_i(x) = s(x_i)\, H_{00}(t) + h_i\, s'(x_i)\, H_{10}(t) + s(x_{i+1})\, H_{01}(t) + h_i\, s'(x_{i+1})\, H_{11}(t), \qquad t = \frac{x-x_i}{h_i}.$$

The global basis functions are

$$\phi_{i,0}(x) = \begin{cases} H_{01}\left(\dfrac{x-x_{i-1}}{h_{i-1}}\right), & x_{i-1} \le x \le x_i, \\[2ex] H_{00}\left(\dfrac{x-x_i}{h_i}\right), & x_i \le x \le x_{i+1}, \\[2ex] 0, & \text{otherwise,} \end{cases}$$

and

$$\phi_{i,1}(x) = \begin{cases} h_{i-1}\, H_{11}\left(\dfrac{x - x_{i-1}}{h_{i-1}}\right), & x_{i-1} \le x \le x_i, \\ h_i\, H_{10}\left(\dfrac{x - x_i}{h_i}\right), & x_i \le x \le x_{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

At endpoints x_0 and x_n, only the relevant half-interval expression applies.

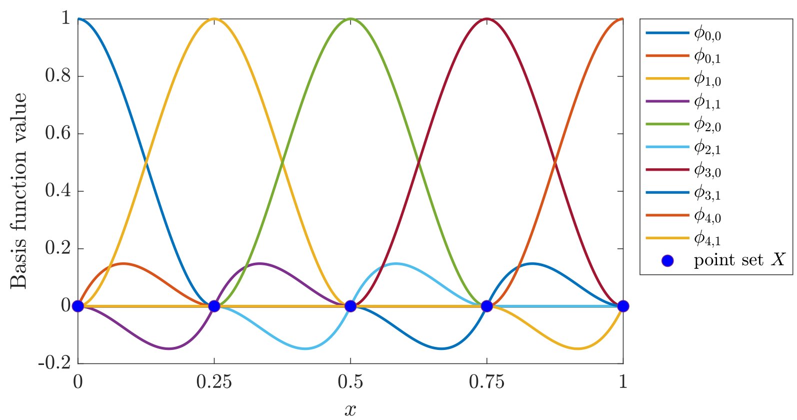

Figure 3: Hermite functions form a basis for spline space $\mathbb{S}_X^{3,1}$ (the C^1 cubic spline space) where $X = \{0, 0.25, 0.5, 0.75, 1\}$.

Figure 3 displays the ten Hermite basis functions spanning the C^1 cubic spline space $\mathbb{S}_X^{3,1}$ defined over the knot set $X = \{0, 0.25, 0.5, 0.75, 1\}$. Although these basis functions are compactly supported and form a complete basis for the space, some of them take negative values, which can introduce numerical instability in practical computations. These basis functions are commonly employed in the finite element method (FEM) for the numerical solution of PDEs. This approach is often referred to as the Hermite finite element method (Hermite FEM).

3.3 B-spline basis

B-splines are among the most fundamental and widely used basis functions in spline theory. They possess *minimal support*, form a *partition of unity*, and are *nonnegative* over their support intervals. These properties make them exceptionally well-suited for numerical computation, geometric modeling, and approximation theory [2].

Definition 3.5 (Curry–Schoenberg representation [1])**.** Let

$$T = \{\ldots, t_0, t_1, t_2, \ldots\}$$

be a nondecreasing sequence of real numbers, called a *knot sequence*. The B-spline of degree ℓ associated with the subsequence $\{t_j, t_{j+1}, \ldots, t_{j+\ell+1}\}$ is defined by

$$B_j^\ell(x) = (t_{j+\ell+1} - t_j)\,[t_j, t_{j+1}, \ldots, t_{j+\ell+1}]\,(x - t)_+^\ell,$$

where $[t_j, t_{j+1}, \ldots, t_{j+\ell+1}]$ denotes the $(\ell+1)$-point *divided difference operator*, acting on the variable t of the one-sided power function $(x-t)_+^\ell$.

This definition remains valid even when some of the knots coincide. In that case, the divided differences are interpreted in the *generalized sense*, i.e., when a knot t_r has multiplicity m, the repeated entries in the divided difference correspond to derivatives:

$$[t_r, \ldots, t_r]f = \frac{f^{(m-1)}(t_r)}{(m-1)!}.$$

Hence, the Curry–Schoenberg representation provides a unified framework for both *uniform* and *nonuniform* B-splines, as well as for spline spaces with *repeated knots* and reduced smoothness.

Cox–de Boor recursive representation

An alternative and computationally efficient definition of B-splines is given by the *Cox–de Boor recursion formula* [2]. This recursive formulation is equivalent to the Curry–Schoenberg representation, but it is more suitable for numerical evaluation and algorithmic construction of spline functions.

Theorem 3.6 (Cox–de Boor recursion formula)**.** Let $T = \{\ldots, t_0, t_1, t_2, \ldots\}$ be a nondecreasing knot sequence. The B-splines of degree ℓ can be represented recursively as follows:

$$\begin{aligned} B_j^0(x) &= \begin{cases} 1, & \text{if } t_j \le x < t_{j+1}, \\ 0, & \text{otherwise,} \end{cases} \\ B_j^\ell(x) &= \frac{x-t_j}{t_{j+\ell}-t_j}\, B_j^{\ell-1}(x) + \frac{t_{j+\ell+1}-x}{t_{j+\ell+1}-t_{j+1}}\, B_{j+1}^{\ell-1}(x), \qquad \ell \ge 1. \end{aligned} \tag{3.4}$$

Here, the convention is that any term with a zero denominator is interpreted as zero. That is, if $t_{j+\ell} = t_j$, the corresponding fraction is defined to be zero.

Proof. The Cox–de Boor recursion can be derived directly from the Curry–Schoenberg representation by applying the recursive properties of divided differences. Specifically, the divided difference of the truncated power function satisfies

$$[t_j, \ldots, t_{j+\ell+1}](x-t)_+^\ell = \frac{x-t_j}{t_{j+\ell}-t_j}\,[t_j, \ldots, t_{j+\ell}](x-t)_+^{\ell-1} + \frac{t_{j+\ell+1}-x}{t_{j+\ell+1}-t_{j+1}}\,[t_{j+1}, \ldots, t_{j+\ell+1}](x-t)_+^{\ell-1}.$$

Multiplying both sides by $(t_{j+\ell+1}-t_j)$ yields precisely the recursive form of $B_j^\ell(x)$ in terms of $B_j^{\ell-1}(x)$ and $B_{j+1}^{\ell-1}(x)$.

□

This recurrence expresses each B-spline of degree ℓ as a linear combination of two adjacent B-splines of degree $\ell-1$, weighted by linear functions in x. The recursion starts from the piecewise constant functions $B_j^0(x)$ and builds higher-degree B-splines incrementally.

Non-negativity, compactly supported and partition of unity properties

We can also prove that each $B_j^\ell(x)$ is nonnegative and has compact support:

$$\mathrm{supp}(B_j^\ell) = [t_j, t_{j+\ell+1}].$$

On the open interval $(t_j, t_{j+\ell+1})$, $B_j^\ell(x)$ is a piecewise polynomial of degree ℓ. It vanishes identically outside its support, and the supports of adjacent B-splines of the same degree overlap only over $\ell + 1$ consecutive knot intervals.

Furthermore, for any $x \in \mathbb{R}$, the collection of B-splines satisfies the *partition of unity* property:

$$\sum_j B_j^\ell(x) = 1.$$

These properties, together with their nonnegativity, make B-splines an ideal basis for stable numerical computations.

Repeated knots and smoothness reduction

If some consecutive knots coincide, say $t_r = t_{r+1} = \cdots = t_{r+m-1}$, the corresponding B-spline loses smoothness at $x = t_r$. Specifically, each repeated knot reduces the continuity order by one. A knot of multiplicity m produces a B-spline that is $C^{\ell-m}$ continuous at that knot. The Cox–de Boor recursion automatically incorporates this effect through its zero-denominator convention, so the same recursion applies without modification.

The Cox–de Boor recursion provides an elegant and computationally robust procedure for constructing B-splines of arbitrary degree and continuity from a given knot sequence. It complements the Curry–Schoenberg representation by offering a numerically stable and incremental mechanism for evaluating spline basis functions.

Construction of the B-spline basis

Let $X = \{x_0, x_1, \ldots, x_n\}$ with $x_0 < x_1 < \cdots < x_n$ be a strictly increasing sequence of distinct knots on $[x_0, x_n]$. To define the B-spline basis functions for the spline space $\mathbb{S}_X^{\ell,k}$ of degree ℓ and smoothness C^k, we first extend the set X by introducing r fictitious knots on each side:

$$X_{\text{ext}} = \{x_{-r}, x_{-r+1}, \ldots, x_0, x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+r}\},$$

where $x_{-r} < \cdots < x_{-1} < x_0$ and $x_n < x_{n+1} < \cdots < x_{n+r}$. These additional points ensure that every B-spline whose support intersects $[x_0, x_n]$ is well-defined over a complete set of $r_{\max} = r + 1$ consecutive intervals. In Theorem 3.7 below we determine the exact value of $r_{\max}$ (and hence r). To impose the desired smoothness C^k at each interior knot, each point of X_{ext} is repeated

$$m = \ell - k$$

times in the final knot vector

$$T = \{\underbrace{x_{-r}, \ldots, x_{-r}}_{m \text{ times}}, \ \ldots, \underbrace{x_0, \ldots, x_0}_{m \text{ times}}, \ \ldots, \ \underbrace{x_{n+r}, \ldots, x_{n+r}}_{m \text{ times}}\}.$$

This repetition ensures that each interior knot has multiplicity m and therefore reduces the continuity at that knot to $C^{\ell-m} = C^k$. Finally, the B-spline basis functions are computed recursively on T using the Cox–de Boor formula (3.4), and those with nonempty support over the physical domain $[x_0, x_n]$ are selected to form the basis for $\mathbb{S}_X^{\ell,k}$. There are exactly $(\ell - k)n + k + 1$ number of such B-splines which equals the dimension of the spline space $\mathbb{S}_X^{\ell,k}$.

Theorem 3.7 (Maximum support of B-spline basis)**.** Let $\mathbb{S}_X^{\ell,k}$ be the spline space of degree ℓ and smoothness C^k at the interior knots $X = \{x_0, x_1, \ldots, x_n\}$, where $0 \leq k \leq \ell - 1$. Assume the corresponding B-spline basis is constructed using a repeated-knot vector

$$T = [\ldots, \underbrace{x_0, \ldots, x_0}_{m \text{ times}}, \underbrace{x_1, \ldots, x_1}_{m \text{ times}}, \ldots, \underbrace{x_n, \ldots, x_n}_{m \text{ times}}, \ldots],$$

with multiplicity $m = \ell - k$. Then each B-spline basis function $B_j^\ell(x)$ has compact support contained within at most

$$r_{\max} = 1 + \left\lfloor \frac{\ell}{\ell - k} \right\rfloor$$

distinct intervals of X.

Proof. A B-spline of degree ℓ is defined by $\ell + 2$ consecutive knots in the repeated sequence T. That is, for some index j,

$$B_j^\ell(x) \quad \text{is supported on} \quad [t_j,\, t_{j+\ell+1}].$$

The vector T is obtained by repeating each distinct knot value x_i exactly m times. To determine the number of distinct x_i's (and thus the number of distinct X-intervals) that can appear within any block of $\ell + 2$ consecutive entries of T, we count how many distinct knots can be covered by such a block.

To maximize the number of distinct knots covered by a block of length $\ell + 2$, we should start the block at the *last copy* of some repeated knot value. In that case, we need only one entry from the first knot group, all m entries from each of the next $q - 2$ full groups, and at least one entry from the last group. Hence, the minimal number of entries needed to cover q distinct knots is

$$N(q) = 1 + (q - 2)m + 1 = (q - 2)m + 2.$$

A B-spline block of length $L = \ell + 2$ can contain at most $q_{\max}$ distinct knots, where $N(q_{\max}) \leq L$. That is,

$$(q_{\max} - 2)m + 2 \leq \ell + 2.$$

Solving for $q_{\max}$ gives

$$q_{\max} \leq \frac{\ell}{m} + 2, \qquad q_{\max} = \left\lfloor \frac{\ell}{m} \right\rfloor + 2.$$

If the B-spline involves $q_{\max}$ distinct knots, its support spans $r_{\max} = q_{\max} - 1$ distinct X-intervals. Substituting $m = \ell - k$ yields

$$r_{\max} = \left\lfloor \frac{\ell}{\ell - k} \right\rfloor + 1.$$

□

Example 3.8. As we pointed out earlier, the spline space $\mathbb{S}_X^{3,1}$ (the cubic C^1 spline space) has dimension $\dim \mathbb{S}_X^{3,1} = 2n+2$. The corresponding B-spline basis is constructed over the knot vector

$$T = \{x_{-1}, x_{-1},\, x_0, x_0,\, \dots,\, x_n, x_n,\, x_{n+1}, x_{n+1}\},$$

where each knot is repeated $m = \ell - k = 2$ times. Since the maximum number of distinct intervals covered by a single basis function is $r_{\max} = 1 + \lfloor \ell/(\ell-k) \rfloor = 1 + \lfloor 3/2 \rfloor = 2$, each B-spline has support over at most two (in this case exactly two) consecutive subintervals. Figure 4 illustrates the B-spline basis functions for this space with $X = \{0,\, 0.25,\, 0.5,\, 0.75,\, 1\}$. As expected, there are $m = 2$ B-spline functions associated with each knot, and all interior B-splines have support spanning two adjacent subintervals, while the first two and last two basis functions are truncated near the domain boundaries.

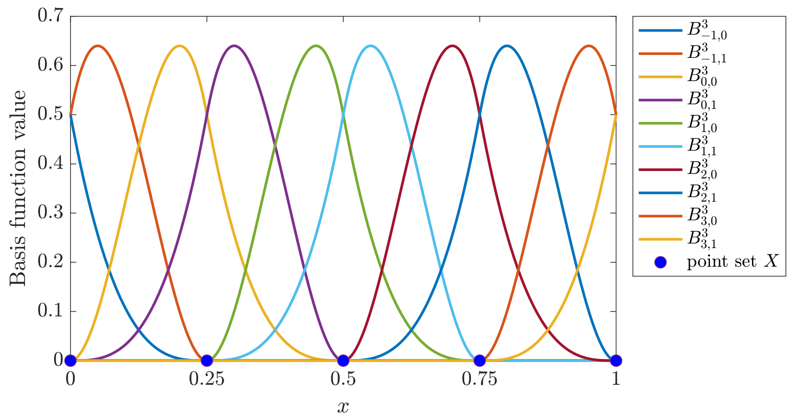

Figure 4: B-spline bases for spline space $\mathbb{S}_X^{3,1}$ (the C^1 cubic spline space) where $X = \{0,\, 0.25,\, 0.5,\, 0.75,\, 1\}$.

Example 3.9. The spline space $\mathbb{S}_X^{4,2}$ (the quartic C^2 spline space) has dimension $\dim \mathbb{S}_X^{4,2} = 2n+3$. The corresponding B-spline basis is constructed over the knot vector

$$T = \{x_{-2}, x_{-2},\, x_{-1}, x_{-1},\, x_0, x_0,\, \dots,\, x_n, x_n,\, x_{n+1}, x_{n+1},\, x_{n+2}, x_{n+2}\},$$

where each knot is repeated $m = \ell - k = 2$ times. The maximum number of distinct intervals covered by a single B-spline is $r_{\max} = 1 + \left\lfloor \frac{\ell}{\ell-k} \right\rfloor = 1 + \left\lfloor \frac{4}{2} \right\rfloor = 3$, so each basis function has support spanning up to three consecutive subintervals.

Figure 5 shows the B-spline basis for this space with $X = \{0,\, 0.25,\, 0.5,\, 0.75,\, 1\}$. As observed, the interior basis functions have supports covering either two or three subintervals, while the first three and last three basis functions are truncated near the domain boundaries.

How do we select the $(\ell - k)n + k + 1$ B-spline basis functions corresponding to the space $\mathbb{S}_X^{\ell,k}$? Recall that, for the extended knot set X_{ext}, there are exactly $m = \ell - k$ B-spline functions associated with each node $x_j \in X_{\text{ext}}$. We denote these by $B^{\ell}_{j,i}$, where $i = 0, 1, \dots, m-1$. Among these functions, those corresponding to indices $j > n-1$ have support entirely outside the physical interval $[x_0, x_n]$ and are therefore discarded. The B-splines associated with $j = 0, 1, \dots, n-1$ remain, yielding a total of $mn = (\ell - k)n$

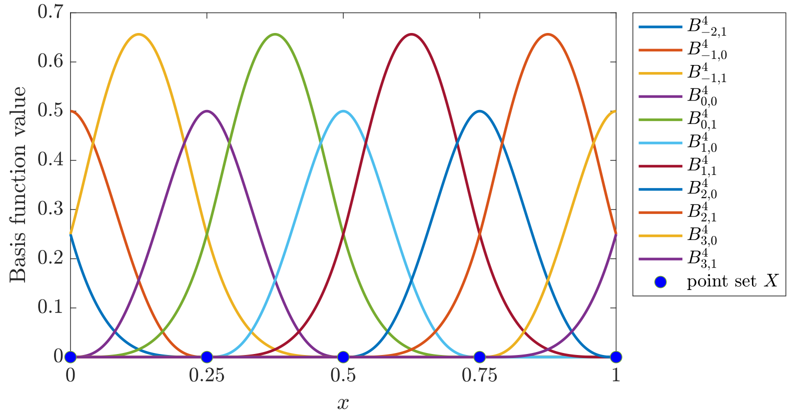

Figure 5: B-spline bases for spline space $\mathbb{S}_X^{4,2}$ (the C^2 quartic spline space) where $X = \{0,\ 0.25,\ 0.5,\ 0.75,\ 1\}$.

basis functions. The remaining $k+1$ functions are taken from the B-splines with indices $j = -1, -2, \ldots, -r$, whose supports partially overlap the domain $[x_0, x_n]$ (i.e., they are supported on the initial subintervals). Together, these form the complete set of $(\ell-k)n+k+1$ basis functions spanning the spline space $\mathbb{S}_X^{\ell,k}$.

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