High-order RBF methods for solving conservation laws

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October 11, 2024

1 Introduction

Many physics and engineering problems give rise to time-dependent partial differential equations (PDEs), and one of the most significant and extensively used among them is the hyperbolic conservation law. One characteristic aspect of conservation laws is the potential development of discontinuities, known as shocks, or sharp fronts in the solution, even when the initial condition is smooth. These discontinuities can lead to non-physical oscillations in the numerical solution. The goal of numerical techniques is to accurately capture these discontinuities while avoiding oscillations and achieving solutions with a high order of convergence. Over the past decades, the finite volume method (FVM) has become a widely-used numerical technique for solving conservation laws. FVM has a rich history and has evolved into various forms and methodologies. These methods are either linear or non-linear and share similarities with finite difference schemes for hyperbolic PDEs, but are also adaptable to unstructured grids. According to the Godunov theorem, linear schemes must be either first-order accurate or oscillatory. To achieve higher accuracy without oscillations, non-linear schemes have been developed. These schemes typically employ the Godunov's approach, an artificial viscosity, or a high-order reconstruction.

2 Background

The earliest attempts at achieving higher than first-order reconstructions dates back to the application of flux and slope limiter methods to obtain second-order accurate schemes in FVM [1]. Subsequently, higher-order essentially non-oscillatory (ENO) reconstructions were developed and widely used for approximating hyperbolic PDEs for computational fluid dynamic problems [3]. ENO reconstruction involves determining sets of stencils surrounding a control volume, computing a reconstruction on each stencil, and selecting the smoothest reconstruction for the control volume. Weighted ENO (WENO) reconstruction improves upon this by using a weighted sum of different reconstructions based on their smoothness.

The ENO and WENO reconstructions were initially based on polynomial interpolation on structured grids (FVM cells). However, for unstructured meshes, these polynomial interpolations can be replaced by *radial basis function (RBF)* approximations, which offer the advantage that are well-suited for scattered data on different geometries and dimensions.

Recently, a weighted smoothed reconstruction (WSR) approach has been proposed in [2] which employs a single central stencil alongside a smoothed RBF reconstruction to suppress non-physical oscillations near shocks or sharp fronts.

3 Project description

Consider a system of conservation law on an open and bounded computational domain $\Omega \subset \mathbb{R}^d$ with an initial condition as follows

$$\frac{\partial u}{\partial t} + \nabla \cdot F(u) = 0, \qquad u\,(0, x) = u_0(x), \tag{1}$$

where $u \equiv u\,(t, x) : I \times \Omega \longrightarrow \mathbb{R}^m$ is the vector of conserved variables (the solution) and $F(u)$ is a *flux* function. Moreover, $I := (0, t_f]$ is a time interval with a final time t_f, and $u_0(x)$ is the initial function.

We solve this system using the finite volume method by employing RBF reconstructions to achieve high-order and non-oscillatory solutions. For example, consider the Kurganov-Petrova-Popov (KPP) equation, which represents a scalar conservation law (with $m = 1$) in 2D (with $n = 2$) and non-convex flux function $F(u) = (\sin u, \cos u)$. The solution to this equation is a rotating wave, which is challenging to approximate numerically. A numerical solution using the WSR method is shown on the left side of Figure 1.

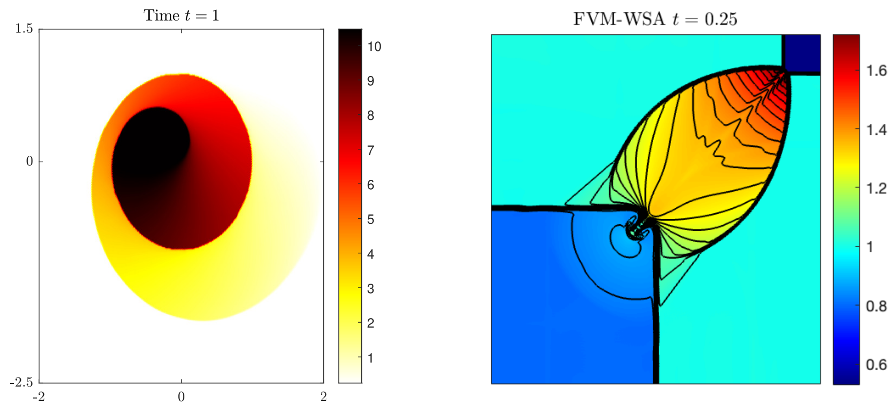

Figure 1: Caption

Another example is the 2D Euler equations for gas dynamics (compressible fluids), where the conserved variables are $(\rho, \rho u_1, \rho u_2, E)$, representing density, momentum, and total energy, respectively. The flux function is given by:

$$F(u) = \begin{bmatrix} \rho u_1 & \rho u_2 \\ \rho u_1^2 + p & \rho u_1 u_2 \\ \rho u_1 u_2 & \rho u_2^2 + p \\ u_1(E + p) & u_2(E + p) \end{bmatrix}$$

where p is the pressure. The numerical solution for the density ρ using 151,768 cells ($h \approx 0.005$) was computed with the WSR approach and is shown on the right-hand side of Figure 1.

In this project, we build on our previous work by applying these methods to other systems of conservation laws, such as Maxwell's equations in electromagnetism. Additionally, we aim to enhance the implementation of the WSR method by introducing parallelization and improving both the stability and the computational efficiency of the involved local linear systems.

4 Planed tasks and required expertise

Planed tasks are:

- Learning and understanding the physics and mathematics of conservation laws
- Learning and understanding the RBF approximation,
- Learning and understanding the FVM for hyperbolic PDEs,
- Implementation of WENO and WSR algorithms for different systems using available codes
- Enhancing the implementation of WSR algorithm

The required backgrounds are:

1. Basic understanding of interpolation techniques,
2. Basic understanding of numerical methods for PDEs,
3. Computer programming with basic understanding of HPC
4. Willingness to learn about numerical PDE solvers.

References

[1] Jan S. Hesthaven, Numerical Methods for Conservation Laws: From Analysis to Algorithms, SIAM, 2018.

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[3] C.W. Shu, Essentially non-oscillatory and weighted essentially non-oscillatory schemes, Acta Numer. 29 (2020) 701-762.